DEGENERATE SIMILARITY SOLUTIONS IN DYNAMIC PROBLEMS ON THE CALCULATION OF STRAINS IN NONLINEARLY ELASTIC BODIES

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We will assume that the moton of a continuum is described by a differential equation in partial derivatives which is dependent on the space coordinate x and time t. If the motion is self-similar, then the solution of the equation has the structure $t^{\alpha}\Phi(\eta)$, where $\Phi(\eta)$ is a function of the dimensionless variable $\eta = x/(at^{\beta})$, a is a dimensional constant, and α and β are exponents dependent on the initial values, the structure of the differential equation, and the boundary and initial conditions. Similarity solutions can be found in problems in which the initial data contains a minimal number of quantities with independent dimensions. This includes problems on the motion of infinite media not having a characteristic linear dimension. This eliminates problems of practical importance in which the medium has finite dimensions, the only exception being the special case of self-similar motions in which β vanishes. We will refer to the similarity solutions corresponding to this case as degenerate. Narrowing of the value of β to zero imposes additional limitations on the initial parameters of the problem. However, this is offset by the possibility of obtaining exact solutions when studying dynamic deformations of nonlinearly elastic bodies of finite dimensions. This is illustrated in the solution of the problem of the bending of beams made of a material for which a power relationship exists between the stresses and strains.

The bending of the beam is described by the differential equation

$$\partial^2 M/\partial x^2 + m\partial^2 w/\partial t^2 = q. \tag{1}$$

Here, w is the deflection; M is the bending moment; m is the linear mass of the beam; q is the linear load. The moment M is connected with the curvature $\partial^2 w/\partial x^2$ by the power relation

$$M = M_0|\partial^2 w/\partial x^2|^{\mu} \operatorname{sign}(\partial^2 w/\partial x^2), \tag{2}$$

where μ is an assigned exponent; M_0 is a dimensional constant. We seek the similarity solution of system (1)-(2) in the form

$$w = w_* n_1 t^{\alpha}\varphi(\xi),\ \xi = x/(bn_2 t^{\beta}); \tag{3}$$

$$M = M_* n_3 t^{\delta}\psi(\xi). \tag{4}$$

Here, w^*, M^*, and b are dimensional constants; α, β, δ, n_1, n_2, n_3 are exponents and numerical multipliers yet to be determined; $\varphi(\xi)$ and $\psi(\xi)$ are dimensionless functions of the variable ξ.

We are examining two variants of beam loading: with a distributed load

$$q = q_* t^{\omega} f(\xi) \tag{5}$$

and with a concentrated force applied in the section x = 0,

$$P = P_* t^{\lambda}, \tag{6}$$

where q^* and P^* are assigned dimensional quantities; f (ξ) is a function of the variable ξ; ω and λ are assigned exponents. Equation (6) makes it possible to write the boundary conditions at x = 0

$$\partial M/\partial x = 0.5 P_* t^{\lambda}. \tag{7}$$

Considering the dimensions of the quantities, we can express the dimensional coefficients in Eqs. (3) and (4) through the assigned quantities. If the load (5) is specified, then

$$w_* = q_* m^{-1},\ b = (M_0 m^{-\mu} q_*^{\mu-1})^{1/(2\mu+2)},\ M_* = (M_0 m^{-\mu} q^{2\mu})^{1/(\mu+1)}, \tag{8}$$

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while if (6) is given, then

$$w_* = (M_0^{-1} P_*^{2(1+\mu)} m^{-(1+2\mu)})^{1/(1+3\mu)}, \; b = (M_0 P_*^{\mu-1} m^{-\mu})^{1/(1+3\mu)}, \tag{9}$$
$$M_* = P_* b.$$

We insert Eqs. (3-5) into (1), (2), and (7) to obtain equations determining the functions $\varphi(\xi)$ and $\psi(\xi)$. We take Eqs. (8) or (9) into account in making this substitution. The transformations made are analogous to those described in [3]. In particular, so that the equation is not explicitly dependent on time, the exponents with t are assumed to be zero in the substitution. This makes the relations

$$\alpha - 2 - \delta + 2\beta = 0, \; \mu(\alpha - 2\beta) - \delta = 0, \; \omega - \delta + 2\beta = 0, \tag{10}$$
$$\lambda - \delta + \beta = 0.$$

In the case where the load (5) is given, we use the first three equations of (10) to determine the values of α, β, and δ. When only (6) is in force, the third equation is excluded from (10).

The equations which determine the functions $\varphi(\xi)$ and $\psi(\xi)$ also include combinations of the arbitrary quantities n_1, n_2, and n_3. They are chosen so as to simplify the values of the coefficients of these equations. The choice can be such as to make some of the coefficients equal to unity. In the case being examined, we take $n_1 = [\alpha(\alpha - 1)]^{-1}$, $n_2 = [\alpha(\alpha - 1]^{-\mu/(2\mu + 2)}$, $n_3 = [\alpha(\alpha - 1)]^{-\mu/(\mu + 1)}$.

We will henceforth use only degenerate solutions, corresponding to $\beta = 0$. With $\beta = 0$, we finally obtain the following differential equations for the functions $\varphi(\xi)$ and $\psi(\xi)$

$$\psi'' + \varphi = f(\xi), \; \psi = |\varphi''|^{\mu} \operatorname{sign} \varphi''. \tag{11}$$

Condition (7) gives the relation

$$\psi'(0) = n_2/(2n_3). \tag{12}$$

With $\beta = 0$, Eqs. (10) lead to

$$\alpha = 2/(1 - \mu), \; \delta = 2\mu/(1 - \mu), \; \omega = 2\mu/(1 - \mu), \; \lambda = 2\mu/(1 - \mu). \tag{13}$$

The last two equations in these formulas link the coefficient μ – characterizing the elastic properties of the material – and the exponents ω and λ – determining the change in the external load in the degenerate solution.

System (11), describing the degenerate similarity solution, makes it possible to calculate beams of finite length. In essence, the variable in this system is the linear coordinate, since the case $\beta = 0$ corresponds to $\xi = x/(bn_2)$. With any fixed value of x (or ξ), we can formulate boundary conditions for the functions φ or ψ or their derivatives, expressed through physical quantities characterizing the bending of beams. The degenerate solution will correspond to changes in the mode of bending of the beam over time, the mode being determined by the form of the function φ.

System (11) was used to calculate the bending of simply supported and rigidly fastened beams under the influence of a uniformly distributed load (5) ($f(\xi) = 1$) or a concentrated force (6) applied to the center of the beam for different spans and two values of μ (1/3 and 0.1). In accordance with (13), each degenerate value of μ in the degenerate solution corresponds to certain values of ω or λ, which in turn determine the character of the change in load over time.

When only the force (6) is given, the function $f(\xi)$ in (11) is set equal to zero. The length of the beam is designated by ℓ, while the origin of the coordinates is taken at the middle of the beam. Boundary conditions at the origin ($\xi = 0$) and at the right support ($\xi = \xi_*$) can be written for the functions in (11). When a concentrated force is acting, one of the conditions with $\xi = 0$ is determined by Eq. (12).

Calculations were performed on an "Élektronika-60" computer by the method of reduction to a Cauchy problem [4]. In accordance with this method, with $\xi = 0$, we used two known boundary conditions for the functions φ, ψ, and their derivatives and arbitrarily assigned two other conditions. We then solved the Cauchy problem. The values of the conditions with $\xi = 0$ were corrected in accordance with the degree of non-closure of the solution at the end of the beam ($\xi = \xi_*$). The process was repeated until the required accuracy was obtained.

Some of the results are shown in Figs. 1-4 in the form of graphs of φ, ψ, ψ', φ', φ'', characterizing the corresponding physical quantities: deflection, bending moment, shearing force, angle of the slope the elastic line, and beam curvature. The graphs were constructed in the

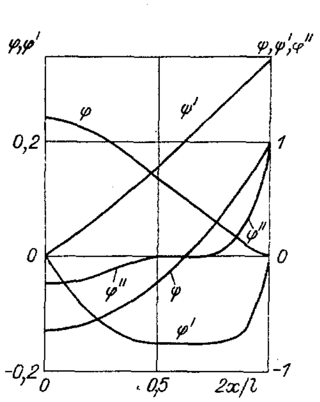

Fig. 1

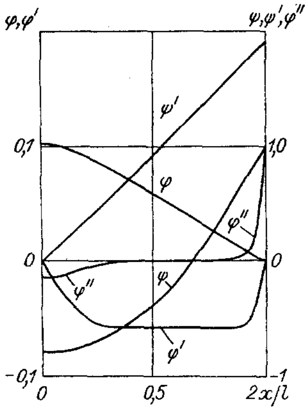

Fig. 2

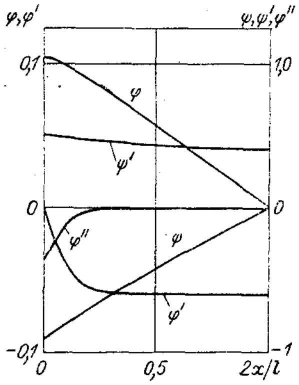

Fig. 3

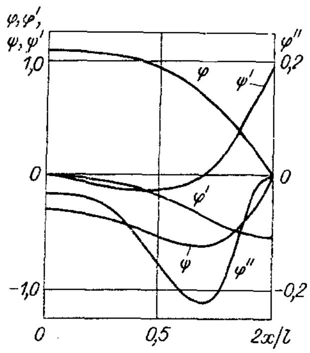

Fig. 4

coordinate $\xi/\xi^* = 2x/\ell$. The parameter $\xi^* = \ell/(2bn_2)$ characterizes the relative length of the beam. The graphs are shown for one half of the beam.

Figure 1 shows the bending of a rigidly fastened beam under a uniformly distributed load for $\mu = 1/3$ and $\xi_* = 2$. Figure 2 shows the results for $\mu = 0.1$, with all of the other parameters retaining their former values. Figure 3 shows results for a simply supported beam loaded in the middle of the span by a concentrated force with $\mu = 0.1$ and $\xi_* = 2$. It is possible to follow the localization of the strains (the increase in curvature φ'') at the most heavily stressed sites of the beam. This is due to adopted relation (2), according to which a small increase in bending moment can correspond to a large increase in strains at $0 < \mu < 1$. Straight beam sections are seen at sites where the stresses are lower.

For long, simply supported beams ($\xi_* \geq 5$) under a uniformly distributed load, the maximum of the bending moment is shifted from the center toward the ends of the beam. This is shown in Fig. 4 for $\mu = 1/3$, $\xi_* = 5$. For long, rigidly fastened beams loaded by a uniformly distributed load, the bending moment has its maximum value at the support, while in the span the maximum moment is also displaced from the center toward the supports.

It should be noted that as μ approaches zero, the coefficients ω and λ also vanish, in accordance with (13). This case corresponds to the action of an instantaneously applied constant load on a beam made of an ideal plastic-rigid material.

A degenerate solution can be found in the problem of the propagation of longitudinal waves. Using the relations presented in [5], it is possible to find a solution for lengthwise impact against the end of a nonlinearly elastic rod having the other end free or fixed. Here, the solution is written in quadratures and is expressed through elliptic integrals.

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TWO-DIMENSIONAL INVERSE PROBLEM OF NONLINEAR ELASTICITY THEORY FOR A HARMONIC MATERIAL

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For a material of harmonic type [1] we consider a two-dimensional inverse problem of nonlinear elasticity theory concerned with the determination of the contour of a hole having uniform strength. This problem was solved in [2] in the linear classical case.

1. Let us assume that the nonlinearly elastic medium under consideration here occupies the plane of the variable z = x + iy, weakened by a curvilinear hole. We assume also that constant normal stresses are applied to the contour L of this hole [3]:

$$\sigma_n = P_0,\ \tau_n = 0, \tag{1.1}$$

and that there is a biaxial tension along the coordinate axes at infinity:

$$\sigma_x^{(\infty)} = P_1,\ \sigma_y^{(\infty)} = P_2. \tag{1.2}$$

Subject to these conditions, we wish to find the shape and location of the contour L so that the tangential stress σ_t will be constant at all of its points:

$$\sigma_t = \sigma \tag{1.3}$$

(σ is constant but unknown).

To solve the problem we make use of complex representations for the stress and deformation fields in terms of functions $\varphi(z)$ and $\psi(z)$, analytic in the physical domain S under consideration (see [4, 5]):

$$\sigma_x + \sigma_y + 4\mu = \frac{\lambda + 2\mu}{\sqrt{J}} q\Omega(q), \tag{1.4}$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = -\frac{4(\lambda + 2\mu)}{\sqrt{J}} \frac{\Omega(q)}{q} \frac{\partial z^*}{\partial z} \frac{\partial z^*}{\partial \bar{z}};$$

$$\frac{\partial z^*}{\partial z} = \frac{\mu}{\lambda + 2\mu} \varphi'^2(z) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\varphi'(z)}{\overline{\varphi'(z)}},\ \frac{\partial z^*}{\partial \bar{z}} = -\frac{\lambda + \mu}{\lambda + 2\mu}\left[\frac{\varphi(z)\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} - \overline{\psi'(z)}\right]; \tag{1.5}$$

$$\sqrt{J} = \frac{\partial z^*}{\partial z}\frac{\partial \bar{z}^*}{\partial \bar{z}} - \frac{\partial z^*}{\partial \bar{z}}\frac{\partial \bar{z}^*}{\partial z},\ q = 2\left|\frac{\partial z^*}{\partial z}\right|,\ \Omega(q) = q - \frac{2(\lambda + \mu)}{\lambda + 2\mu} \tag{1.6}$$

(λ, μ are the Lamé elastic constants). For large |z| these functions have the asymptotics

$$\varphi(z) = a_0 z + O(z^{-1}),\ \psi(z) = b_0 z + O(z^{-1}) \tag{1.7}$$

(a_0, and b_0 are known constants [6]);

$$a_0 = \left[\frac{\lambda + \mu}{\mu} \frac{2\mu(P_1 + P_2) + P_1P_2 + 4\mu^2}{\lambda(P_1 + P_2) - P_1P_2 + 4\mu(\lambda + \mu)}\right]^{1/2}, \tag{1.8}$$

$$b_0 = \frac{(\lambda + 2\mu)(P_1 - P_2)}{\lambda(P_1 + P_2) - P_1P_2 + 4\mu(\lambda + \mu)}.$$

Comparing the relations (1.4), we obtain the equation

$$\frac{\partial \bar{z}^*}{\partial z} = \frac{\sigma_x - \sigma_y - 2i\tau_{xy}}{\sigma_x + \sigma_y + 4\mu} \frac{\partial z^*}{\partial z}, \tag{1.9}$$

using this, we have, based on relations (1.4)-(1.6), after some calculations,

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